Spherical harmonic polynomials for higher bundles

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Abstract

We give a method of decomposing bundle-valued polynomials compatible with the action of the Lie group $Spin(n)$, where important tools are $Spin(n)$-equivariant operators and their spectral decompositions. In particular, the top irreducible component is realized as an intersection of kernels of these operators.

0 Introduction

Spherical harmonic polynomials or spherical harmonics are polynomial solutions of the Laplace equation $\Box\phi(x) = \sum \partial^2\phi/\partial x_i^2 = 0$ on $\mathbf{R}^n$. These are fundamental and classical objects in mathematics and physics. It is natural that we consider vector-valued spherical harmonic polynomials. For example, the polynomial solutions of the Dirac equation $D\phi(x) = 0$ on $\mathbf{R}^n$ are studied in Clifford analysis (see [6], [8], and [14]). They are spinor-valued polynomials and called spherical monogenics. We also have other examples in [5], [7], [9], and [12], where we can give spectral information of some basic operators on sphere. Recently, the first-order $Spin(n)$-equivariant differential operators have been studied like Dirac operator and Rarita-Schwinger operator (see [1]-[5], [10], and [11]). These operators are called higher spin Dirac operators or Stein-Weiss operators. In this paper, we give a method to analyze polynomial sections for natural bundles on $\mathbf{R}^n$ by using higher spin Dirac operators and Clifford homomorphisms. Here, Clifford homomorphism is a natural generalization of Clifford algebra given in [10] and [11].

Let S^q (resp. H^q) be the spaces of polynomials (resp. harmonic polynomials) with degree q on the n-dimensional Euclidean space $\mathbf{R}^n$. We know

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that H^q is an irreducible representation space for $Spin(n)$, and S^q has irreducible decomposition, $\oplus_{0\le k\le[q/2]} r^{2k}H^{q-2k}$, where r is $|x| = \sqrt{x_1^2+\cdots+x_n^2}$. To give such a decomposition, we use the invariant operator $-r^2\Box$ and its spectral decomposition. In particular, the top component H^q is the kernel of the operator $-r^2\Box$. Now, we consider natural irreducible bundle $\mathbf{R}^n \times V_\rho$ on $\mathbf{R^n}$, where V_ρ is an irreducible representation space with highest weight ρ for $Spin(n)$. Our interest is to analyze the space of V_ρ-valued polynomials, $S^q \otimes V_\rho$. For that purpose, we use higher spin Dirac operators $\{D^\rho_{\lambda_k}\}_k$ and algebraic operators $\{x^\rho_{\lambda_k}\}_k$. Then we have an invariant operator E whose spectral decomposition gives the irreducible decomposition of $S^q\otimes V_\rho$ like the operator $-r^2\Box$. In particular, the top irreducible component is the kernel of E and realized as an intersection of kernels of higher spin Dirac operators.

1 Clifford Homomorphisms

In this section, we review Clifford homomorphisms given in [11]. Let $\mathfrak{spin}(n) \simeq \mathfrak{so}(n)$ be the Lie algebra of the spin group $Spin(n)$ or orthogonal group $SO(n)$. The Lie algebra $\mathfrak{spin}(n)$ is realized by using the Clifford algebra Cl_n associated to $\mathbf{R}^n$: we choose the standard basis $\{e_i\}_i$ of $\mathbf{R}^n$ and put $[e_i,e_j] := e_ie_j - e_je_i$ in Cl_n. Then $\{[e_i,e_j]\}_{i,j}$ span the Lie algebra $\mathfrak{spin}(n)$ in Cl_n.

The irreducible finite dimensional unitary representations of $\mathfrak{spin}(n)$ or $Spin(n)$ are parametrized by dominant weights $\rho = (\rho^1,\cdots,\rho^m) \in \mathbf{Z}^m \cup (\mathbf{Z}+1/2)^m$ satisfying that

$$\rho^1 \ge \cdots \ge \rho^{m-1} \ge |\rho^m|, \quad \text{for } n = 2m, \tag{1.1}$$

$$\rho^1 \ge \cdots \ge \rho^{m-1} \ge \rho^m \ge 0, \quad \text{for } n = 2m+1. \tag{1.2}$$

We denote by (π_ρ, V_ρ) not only the representation of $Spin(n)$ but also its infinitesimal one of $\mathfrak{spin}(n)$ with highest weight ρ. When writing dominant weights, we denote a string of j k's for k in $\mathbf{Z}\cup(\mathbf{Z}+1/2)$ by k_j. For example, the adjoint representation $(\mathrm{Ad}, \mathbf{R}^n\otimes\mathbf{C})$ of $Spin(n)$ (resp. $\mathfrak{spin}(n)$) has the highest weight $(1, 0_{m-1})$, where the action is $\pi_{\mathrm{Ad}}(g)u = gug^{-1}$ for g in $Spin(n)$ (resp. $\pi_{\mathrm{Ad}}([e_i,e_j])u := [[e_i,e_j],u]$).

We consider an irreducible representation (π_ρ, V_ρ) and the tensor representation $(\pi_\rho\otimes\pi_{\mathrm{Ad}}, V_\rho\otimes_{\mathbf{C}}\mathbf{R}^n)$. We decompose it to irreducible components, $V_\rho\otimes_{\mathbf{C}}\mathbf{R}^n = \sum_{0\le k\le N} V_{\lambda_k}$. For u in $\mathbf{R}^n$, we have the following bilinear mapping for each k:

$$\mathbf{R}^n \times V_\rho \ni (u,\phi) \mapsto p^\rho_{\lambda_k}(u)\phi := \Pi^\rho_{\lambda_k}(\phi\otimes u) \in V_{\lambda_k}, \tag{1.3}$$

where $\Pi^\rho_{\lambda_k}$ is the orthogonal projection from $V_\rho \otimes_{\mathbf{C}} \mathbf{R}^n$ onto V_{λ_k}. We call the linear mapping $p^\rho_{\lambda_k}(u) : V_\rho \to V_{\lambda_k}$ *the Clifford homomorphism* from V_ρ to V_{λ_k}, and denote by $(p^\rho_{\lambda_k}(u))^*$ the adjoint operator of $p^\rho_{\lambda_k}(u)$ with respect to the inner products on V_ρ and V_{λ_k}. If we consider the spinor representation (π_Δ, V_Δ), then the Clifford homomorphism from V_Δ to itself is the usual Clifford action of $\mathbf{R}^n$ on V_Δ, which satisfy the relation $e_ie_j + e_je_i = -\delta_{ij}$. In general cases, we have a lot of relations among these homomorphisms.

Theorem 1.1 ([11]). *For any non-negative integer q, we define the bilinear mapping r^q_ρ as follows:*

$$\begin{aligned} &r^q_\rho : \mathbf{R}^n \times \mathbf{R}^n \ni (u,v) \mapsto \\ &\quad \left(-\frac{1}{4}\right)^q \sum_{l_1,\cdots,l_{q-1}} \pi_\rho([u,e_{l_1}])\pi_\rho([e_{l_1},e_{l_2}])\cdots\pi_\rho([e_{l_{q-1}},v]) \in \mathrm{End}(V_\rho), \end{aligned} \tag{1.4}$$

and $r^0_\rho(u,v) := \langle u,v\rangle$. Then we have

$$\sum_{0\le k\le N} m(\lambda_k)^q (p^\rho_{\lambda_k}(u))^* p^\rho_{\lambda_k}(v) = r^q_\rho(u,v), \tag{1.5}$$

where $m(\lambda_k)$ is the conformal weight assigned from V_ρ to V_{λ_k}.

In this paper, we will use the case of $q = 0$ and $q = 1$:

$$\sum_{0\le k\le N} (p^\rho_{\lambda_k}(e_j))^* p^\rho_{\lambda_k}(e_i) = \delta_{ij}, \tag{1.6}$$

$$\sum_{0\le k\le N} m(\lambda_k)(p^\rho_{\lambda_k}(e_j))^* p^\rho_{\lambda_k}(e_i) = -\frac{1}{4}\pi_\rho([e_j,e_i]). \tag{1.7}$$

Remark 1.1. The endomorphisms $\{r^q_\rho(e_i,e_j)\}_{i,j}$ are useful to compute the eigenvalues of the higher Casimir operators (see [13] and [15]).

The Clifford homomorphisms also satisfy the following properties.

Proposition 1.2 ([11]). *Let u be in $\mathbf{R}^n$, g in $Spin(n)$, and $[e_i,e_j]$ in $\mathfrak{spin}(n)$. Then we have*

$$p^\rho_{\lambda_k}(gug^{-1}) = \pi_{\lambda_k}(g)p^\rho_{\lambda_k}(u)\pi_\rho(g^{-1}), \tag{1.8}$$

and

$$p^\rho_{\lambda_k}([[e_i,e_j],u]) = \pi_{\lambda_k}([e_i,e_j])p^\rho_{\lambda_k}(u) - p^\rho_{\lambda_k}(u)\pi_\rho([e_i,e_j]). \tag{1.9}$$

2 Invariant operators on polynomials for higher bundles

In the first part of this section, we give a well-known method to decompose the space of complex-valued polynomials on $\mathbf{R}^n$. We denote the canonical coordinate on $\mathbf{R}^n$ by $(x_i, \cdots, x_n)$, and the space of complex-valued polynomials with degree q on $\mathbf{R}^n$ by S^q. The vector space $\sum_q S^q$ has the Hermitian inner product satisfying $(\partial/\partial x_i f(x), g(x)) = (f(x), x_i g(x))$. The polynomial representation $(\pi_s, \sum S^q)$ of $\mathfrak{spin}(n)$ is defined by

$$(\pi_s([e_k, e_l])f)(x) := 4(-x_k \frac{\partial}{\partial x_l} + x_l \frac{\partial}{\partial x_k})f(x). \tag{2.1}$$

To decompose the space $\sum S^q$, we use invariant operators compatible with the action of $\mathfrak{spin}(n)$. When the operator on S^q maps to S^{q-k}, the order of the operator is said to be k. On $\sum S^q$, we have the following invariant operators: the Laplacian operator $\Box := -\sum \partial^2/\partial x_i^2$, and the 0-th order operator $r\partial/\partial r = \sum x_i \partial/\partial x_i$ called the Euler operator, where r^2 is $\sum x_i^2$. The Euler operator measures the degree of polynomials. In other words, the vector space S^q is the eigenspace with eigenvalue q for the operator $r\partial/\partial r$. To decompose S^q further, we use the 0-th order invariant operator $-r^2\Box$. This operator has the spectral decomposition corresponding to the irreducible decomposition. In fact, we show that S^q is isomorphic to $\oplus_{0\leq k\leq [q/2]} r^{2k} H^{q-2k}$ and the eigenvalue of $-r^2\Box$ on $r^{2k}H^{q-2k}$ is $k(2q-2k+n-2)$, where H^q is the space of harmonic polynomials with degree q. In particular, the top component H^q is the kernel of $-r^2\Box$ and has the highest weight $h^q := (q, 0_{m-1})$. Thus, to decompose a representation space into irreducible components, we should investigate the spectral decompositions of invariant operators.

Now, we shall consider the space of polynomials for higher bundles on $\mathbf{R}^n$. Let (π_ρ, V_ρ) be an irreducible unitary representation of $\mathfrak{spin}(n)$. Then we have the (trivial) higher bundle $\mathbf{S}_\rho := \mathbf{R}^n \times V_\rho$, and consider the polynomial sections of $\mathbf{S}_\rho$, that is, the V_ρ-valued polynomials $\sum S^q \otimes V_\rho$. This vector space is a representation space on where more invariant operators exist in addition to $-r^2\Box$ and $r\partial/\partial r$. Here, the action of $\mathfrak{spin}(n)$ on $\sum_q S^q \otimes V_\rho$ is given as the tensor representation:

$$\begin{aligned}\mathfrak{spin}(n) \times S^q \otimes V_\rho \ni ([e_k, e_l], f\otimes\phi) \to \\ 4(-x_k \frac{\partial}{\partial x_l} + x_l \frac{\partial}{\partial x_k})f\otimes\phi + f\otimes \pi_\rho([e_k, e_l])\phi \in S^q \otimes V_\rho.\end{aligned} \tag{2.2}$$

We recall the Clifford homomorphism from V_ρ to V_{λ_k} given in Section 1.

By using the Clifford homomorphism, we introduce the following operators:

$$x^{\rho}_{\lambda_k} := \sum x_i p^{\rho}_{\lambda_k}(e_i) : S^q \otimes V_\rho \to S^{q+1} \otimes V_{\lambda_k}, \tag{2.3}$$

$$(x^{\rho}_{\lambda_k})^* := \sum x_i (p^{\rho}_{\lambda_k}(e_i))^* : S^q \otimes V_{\lambda_k} \to S^{q+1} \otimes V_\rho, \tag{2.4}$$

$$D^{\rho}_{\lambda_k} := \sum p^{\rho}_{\lambda_k}(e_i) \frac{\partial}{\partial x_i} : S^q \otimes V_\rho \to S^{q-1} \otimes V_{\lambda_k}, \tag{2.5}$$

$$(D^{\rho}_{\lambda_k})^* := -\sum (p^{\rho}_{\lambda_k}(e_i))^* \frac{\partial}{\partial x_i} : S^q \otimes V_{\lambda_k} \to S^{q-1} \otimes V_\rho. \tag{2.6}$$

The differential operators $D^{\rho}_{\lambda_k}$ and $(D^{\rho}_{\lambda_k})^*$ are called the higher spin Dirac operators, which are generalization of the Dirac operator for higher bundles. If we define the inner product on $S^q \otimes V_\rho$ by the tensor inner product, then we show that the adjoint operators of $x^{\rho}_{\lambda_k}$ and $(x^{\rho}_{\lambda_k})^*$ are $-(D^{\rho}_{\lambda_k})^*$ and $D^{\rho}_{\lambda_k}$, respectively.

We can show that the above operators are invariant operators on the $\mathfrak{spin}(n)$-module $\sum_q S^q \otimes V_\rho$.

Proposition 2.1. *The operators* (2.3)-(2.6) *are invariant operators.*

Proof. We prove only the invariance of $x^{\rho}_{\lambda_k}$. It follows from the equation (1.9) that we have

$$\begin{aligned}
&(-4x_k \frac{\partial}{\partial x_l} + 4x_l \frac{\partial}{\partial x_k} + \pi_{\lambda_k}([e_k, e_l])) x^{\rho}_{\lambda_k} \\
=&\sum_i (-4x_k \frac{\partial}{\partial x_l} + 4x_l \frac{\partial}{\partial x_k} + \pi_{\lambda_k}([e_k, e_l])) x_i p^{\rho}_{\lambda_k}(e_i) \\
=&\sum_i 4p^{\rho}_{\lambda_k}(e_i)(-\delta_{li} x_k - x_k x_i \frac{\partial}{\partial x_l} + \delta_{ki} x_l + x_l x_i \frac{\partial}{\partial x_k}) \\
&\qquad + x_i \{p^{\rho}_{\lambda_k}(e_i) \pi_\rho([e_k, e_l]) + p^{\rho}_{\lambda_k}([[e_k, e_l], e_i])\} \\
=&4(-p^{\rho}_{\lambda_k}(e_l) x_k + p^{\rho}_{\lambda_k}(e_k) x_l) + x^{\rho}_{\lambda_k} 4(-x_k \frac{\partial}{\partial x_l} + x_l \frac{\partial}{\partial x_k}) \\
&\qquad + x^{\rho}_{\lambda_k} \pi_\rho([e_k, e_l]) + \sum x_i (4\delta_{ki} p^{\rho}_{\lambda_k}(e_l) - 4\delta_{li} p^{\rho}_{\lambda_k}(e_k)) \\
=&x^{\rho}_{\lambda_k}(-4x_k \frac{\partial}{\partial x_l} + 4x_l \frac{\partial}{\partial x_k} + \pi_\rho([e_k, e_l])).
\end{aligned} \tag{2.7}$$

∎

We shall investigate relations among these invariant operators, and reconstruct the Laplacian operator and the Euler operator. First, the formula (1.6) induces the following lemma.

Lemma 2.2. *The invariant operators* (2.3)-(2.6) *satisfy that*

$$\sum_{0\le k\le N} (x^\rho_{\lambda_k})^* x^\rho_{\lambda_k} = \sum_i (x_i)^2 = r^2, \quad \sum_{0\le k\le N} (D^\rho_{\lambda_k})^* D^\rho_{\lambda_k} = \Box, \tag{2.8}$$

$$\sum_{0\le k\le N} (D^\rho_{\lambda_k})^* x^\rho_{\lambda_k} = -n - r\frac{\partial}{\partial r}, \quad \sum_{0\le k\le N} (x^\rho_{\lambda_k})^* D^\rho_{\lambda_k} = r\frac{\partial}{\partial r}. \tag{2.9}$$

In similar way, the formula (1.7) gives the following lemma.

Lemma 2.3. *The invariant operators* (2.3)-(2.6) *satisfy that*

$$\sum_{0\le k\le N} m(\lambda_k)(x^\rho_{\lambda_k})^* x^\rho_{\lambda_k} = 0, \quad \sum_{0\le k\le N} m(\lambda_k)(D^\rho_{\lambda_k})^* D^\rho_{\lambda_k} = 0. \tag{2.10}$$

Remark 2.1. The second equation in (2.10) means that $\mathbf{R}^n$ is a flat space (see [11]).

Since we have already given the decomposition of S^q, we shall decompose the V_ρ-valued harmonic polynomials $H^q \otimes V_\rho$. So we need relations among the Laplacian and the operators (2.3)-(2.6).

Lemma 2.4. *The Laplace operator* $\Box$ *and the operators* (2.3)-(2.6) *satisfy that*

$$[\Box, (D^\rho_{\lambda_k})^*] = 0, \quad [\Box, D^\rho_{\lambda_k}] = 0, \tag{2.11}$$

$$[\Box, x^\rho_{\lambda_k}] = -2D^\rho_{\lambda_k}, \quad [\Box, (x^\rho_{\lambda_k})^*] = 2(D^\rho_{\lambda_k})^*. \tag{2.12}$$

From Lemma 2.3 and 2.4, we have 0-th order invariant operators compatible with the Laplacian $\Box$.

Corollary 2.5. *We consider the* 0*-th order operators* $\sum_k m(\lambda_k)(D^\rho_{\lambda_k})^* x^\rho_{\lambda_k}$ *and* $\sum_k m(\lambda_k)(x^\rho_{\lambda_k})^* D^\rho_{\lambda_k}$*. These operators commute with the Laplace operator:*

$$[\Box, \sum_k m(\lambda_k)(D^\rho_{\lambda_k})^* x^\rho_{\lambda_k}] = [\Box, \sum_k m(\lambda_k)(x^\rho_{\lambda_k})^* D^\rho_{\lambda_k}] = 0. \tag{2.13}$$

Furthermore, these two operators coincide with each other.

Proof. We can easily show that

$$\begin{aligned}
&\sum_k m(\lambda_k)(-(D^\rho_{\lambda_k})^* x^\rho_{\lambda_k} + (x^\rho_{\lambda_k})^* D^\rho_{\lambda_k})\\
=&-\sum_{i,j}(x_j\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial x_j})(\frac{1}{4}\pi_\rho([e_j, e_i]))\\
=&0.
\end{aligned}$$

So we have proved the lemma. ∎

This corollary means that the operator $\sum_k m(\lambda_k)(x^\rho_{\lambda_k})^* D^\rho_{\lambda_k}$ acts on $H^q \otimes V_\rho$ and has a spectral decomposition.

Proposition 2.6. *Let $(\sum_\mu \pi_\mu, \sum_\mu V_\mu)$ be the irreducible decomposition of $(\pi_{h^q} \otimes \pi_\rho, H^q \otimes V_\rho)$. The 0-th order invariant operator $\sum_k m(\lambda_k)(x^\rho_{\lambda_k})^* D^\rho_{\lambda_k}$ has the following spectral decomposition on $H^q \otimes V_\rho$:*

$$\sum_k m(\lambda_k)(x^\rho_{\lambda_k})^* D^\rho_{\lambda_k} = m(\mu, q) \qquad \text{on } V_\mu. \tag{2.14}$$

The constant $m(\mu, q)$ is given by

$$m(\mu, q) := \frac{1}{2}(q^2 + (n-2)q + \|\rho + \delta\|^2 - \|\mu + \delta\|^2), \tag{2.15}$$

where δ is half the sum of positive roots, and $\|\cdot\|$ is the canonical norm on the weight space, that is, $\|\nu\|^2 = \sum_{1 \le i \le m} (\nu^i)^2$.

Proof. We can show that

$$\begin{aligned}
&\sum_k m(\lambda_k)(-(D^\rho_{\lambda_k})^* x^\rho_{\lambda_k} - (x^\rho_{\lambda_k})^* D^\rho_{\lambda_k}) \\
&= -\sum_{ij} (-x_j \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_j})(\frac{1}{4}\pi_\rho([e_j, e_i])) \\
&= -2 \sum_{ij} \frac{1}{32} \pi_{h^q}([e_i, e_j]) \otimes \pi_\rho([e_i, e_j]).
\end{aligned} \tag{2.16}$$

The last equation is realized by using the Casimir operators. In fact, we can show that

$$\sum_{ij} \frac{1}{32} \pi_{h^q}([e_i, e_j]) \otimes \pi_\rho([e_i, e_j]) = C_{h^q \otimes \rho} - C_{h^q} \otimes \mathrm{id} - \mathrm{id} \otimes C_\rho. \tag{2.17}$$

Here, the Casimir operator C_ν on the irreducible representation space V_ν is defined by

$$C_\nu := \frac{1}{64} \sum_{ij} \pi_\nu([e_i, e_j]) \pi_\nu([e_i, e_j]), \tag{2.18}$$

and acts as the constant $-(\|\delta + \nu\|^2 - \|\delta\|^2)/2$ on V_ν. Thus we have proved the proposition. ∎

Instead of the 0-th order operator in the above proposition, we consider the following operator corresponding to the Bochner type Laplacian on the bundle $\mathbf{S}_\rho$ (see [11]):

$$E := \sum_{1 \le k \le N} \left(1 - \frac{m(\lambda_k)}{m(\lambda_0)}\right)(x^\rho_{\lambda_k})^* D^\rho_{\lambda_k}, \tag{2.19}$$

where the weights $\{\lambda_k\}_k$ satisfy that $\lambda_0 > \lambda_1 > \cdots > \lambda_N$ with respect to the lexicographical order on the weight space. This operator E is obtained by eliminateing the top operator $(x^\rho_{\lambda_0})^* D^\rho_{\lambda_0}$ from the equations $\sum_k (x^\rho_{\lambda_k})^* D^\rho_{\lambda_k}$ and $\sum_k m(\lambda_k)(x^\rho_{\lambda_k})^* D^\rho_{\lambda_k}$. Then we have the following theorem.

Theorem 2.7. *Let $(\sum_\mu \pi_\mu, \sum_\mu V_\mu)$ be the irreducible decomposition of $(\pi_{h^q} \otimes \pi_\rho, H^q \otimes V_\rho)$, where $h^q = (q, 0_{m-1})$ and $\rho = (\rho^1, \cdots, \rho^m)$. The 0-th order invariant operator E is a non-negative operator and has the spectral decomposition on $H^q \otimes V_\rho$ as follows:*

$$E = q + \frac{m(\mu, q)}{\rho^1} \qquad \text{on } V_\mu, \tag{2.20}$$

where the constant $m(\mu, q)$ is given in (2.15). In particular, the 0-eigenspace is the irreducible representation space with highest weight $\mu_0 := h^q + \rho$.

In this theorem, we remark that the eigenvalues $\{e(\mu)\}$ of E order as $0 = e(\mu_0) < e(\mu_1) \le e(\mu_2) \le \cdots$ for $\mu_0 > \mu_1 \ge \mu_2 \ge \cdots$. Here, the top component (π_{μ_0}, V_{μ_0}) certainly exists with multiplication one.

Corollary 2.8. *The irreducible representation with highest weight μ_0 in $H^q \otimes V_\rho$ is realized as follows:*

$$V_{\mu_0} = \bigcap_{1 \le k \le N} \ker D^\rho_{\lambda_k}, \tag{2.21}$$

where $\ker D^\rho_{\lambda_k}$ is the kernel of $D^\rho_{\lambda_k}$ on $H^q \otimes V_\rho$.

3 Examples

In this section, we give some examples: spinor-valued harmonic polynomials and p-form-valued harmonic polynomials (see [6], [7]-[9], [12], and [14]).

Example 3.1 (spinor-valued harmonic polynomials). We shall investigate only the odd dimensional case, that is, the case of $n = 2m+1$. Let V_Δ be the spinor

space with highest weight $\Delta = ((1/2)_m)$. We consider the spinor-valued harmonic polynomials $H^q \otimes V_\Delta$, and have invariant operators: the Clifford multiplication $x = -x^* = \sum x_i e_i$ and the Dirac operator $D = D^* = \sum e_i \partial/\partial x_i$, twistor operator T and so on. Then the 0-th order invariant operator E in Theorem 2.7 is $-xD = x^* D$.

Now, we show that $H^q \otimes V_\Delta$ has the irreducible decomposition $V_{\mu_0} \oplus V_{\mu_1}$, where $\mu_0 = h^q + \Delta = (q+1/2, (1/2)_{m-1})$ and $\mu_1 = (q-1/2, (1/2)_{m-1})$. Then we have the spectral decomposition of $-xD$:

$$-xD = \begin{cases} 0 & \text{on } V_{\mu_0} \\ n+2q-2 & \text{on } V_{\mu_1}. \end{cases} \tag{3.1}$$

In particular, we have

$$V_{\mu_0} = \ker D, \qquad V_{\mu_1} = H^q \otimes V_\Delta / \ker D. \tag{3.2}$$

Example 3.2 (p-form-valued harmonic polynomials). Let Λ^p be the exterior tensor product space of $\mathbf{R}^n$ with degree p, which is the irreducible representation space with highest weight $(1_p, 0_{m-p})$. We consider the p-form-valued harmonic polynomials $H^q \otimes \Lambda^p$, and have invariant operators: the exterior derivative $d = \sum e_i \wedge \partial/\partial x_i$, its adjoint $d^* = -\sum i(e_i)\partial/\partial x_i$, the conformal killing operator C, $x_\wedge = \sum x_i e_i \wedge$, and $i(x) = \sum x_i i(e_i)$ and so on. Here, $i(e_i)$ denotes the interior product of e_i. Then we have the spectral decomposition of $E = i(x)d - x_\wedge d^*$ on $H^q \otimes \Lambda^p$:

$$i(x)d - x_\wedge d^* = \begin{cases} 0 & \text{on } V_{\mu_0} \\ q+p & \text{on } V_{\mu_1} \\ n+q-p & \text{on } V_{\mu_2} \\ n+2q-2 & \text{on } V_{\mu_3} \text{ (for } q \geq 2), \end{cases} \tag{3.3}$$

where $\mu_0 = (q+1, 1_{p-1}, 0_{m-p})$, $\mu_1 = (q, 1_p, 0_{m-p-1})$, $\mu_2 = (q, 1_{p-2}, 0_{m-p+1}$, and $\mu_3 = (q-1, 1_{p-1}, 0_{m-p})$. In particular, we have $V_{\mu_0} = \ker d \cap \ker d^*$.

4 Discussion

In the case of p-form-valued harmonic polynomials, we can show that

$$V_{\mu_1} = \ker d / \ker d \cap \ker d^*, \tag{4.1}$$

$$V_{\mu_2} = \ker d^* / \ker d \cap \ker d^*, \tag{4.2}$$

$$V_{\mu_3} = H^q \otimes \Lambda^p / (\ker d + \ker d^*). \tag{4.3}$$

Thus, we can realize the irreducible components by using kernels of d and d^*. In general case, we may realize any irreducible component of $H^q \otimes V_\rho$ by using kernels of higher spin Dirac operators (for the case of Rarita-Schwinger operator, see [5]).

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